THE MANGA GUIDE™ TO
LINEAR ALGEBRA

SUPPLEMENTAL APPENDIXES

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A
WORKBOOK

PROBLEM SETS

PROBLEM SET 1

Let's start off with the 2×2 matrix $\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}$. Use it in the following six problems.

1. Calculate the determinant.
2. Use the formula $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \dfrac{1}{a_{11}\,a_{22} - a_{12}\,a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$ to calculate the inverse.
3. Find the inverse using Gaussian elimination.
4. Find all eigenvalues and eigenvectors.
5. Express the matrix in the form $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{-1}$
6. Solve the linear system of equations $\begin{cases} 4x_1 - 1x_2 = 1 \\ 5x_1 - 2x_2 = -1 \end{cases}$ using Cramer's rule.

PROBLEM SET 2

Next up is the 3×3 matrix $\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$. Use it in the following two problems.

1. Prove that the matrix column vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ are linearly independent (i.e., that the matrix rank is equal to three).
2. Calculate the determinant.

PROBLEM SET 3

Determine whether the following sets are subspaces of R^3.

1. $\left\{ \begin{pmatrix} \alpha \\ \beta \\ 5\alpha - 7\beta \end{pmatrix} \middle| \begin{array}{l} \alpha \text{ and } \beta \text{ are arbitrary} \\ \text{real numbers} \end{array} \right\}$

2. $\left\{ \begin{pmatrix} \alpha \\ \beta \\ 5\alpha - 7 \end{pmatrix} \middle| \begin{array}{l} \alpha \text{ and } \beta \text{ are arbitrary} \\ \text{real numbers} \end{array} \right\}$

NOTE *Have a look at Appendixes C and D before trying problem set 4.*

PROBLEM SET 4

Let's deal with the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ for the next set of problems.

1. Calculate the distance to the origin for both vectors.
2. Calculate the scalar product of the two vectors.
3. Calculate the angle between the two vectors.
4. Calculate the cross product of the two vectors.

SOLUTIONS

PROBLEM SET 1

1. $\det\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} = 4 \cdot (-2) - (-1) \cdot 5 = -8 + 5 = -3$

2. $\dfrac{1}{4 \cdot (-2) - (-1) \cdot 5}\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} = \dfrac{1}{-3}\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

3. Here is the solution:

$$\begin{pmatrix} 4 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{pmatrix}$$

Multiply row 1 by 2 and subtract row 2 from row 1.

$$\begin{pmatrix} 3 & 0 & 2 & -1 \\ 5 & -2 & 0 & 1 \end{pmatrix}$$

Multiply row 1 by 5 and row 2 by 3. Subtract row 1 from row 2.

$$\begin{pmatrix} 15 & 0 & 10 & -5 \\ 0 & -6 & -10 & 8 \end{pmatrix}$$

Divide row 1 by 15 and row 2 by −6.

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} & -\frac{4}{3} \end{pmatrix}$$

4. The eigenvalues are roots of the characteristic equation

$$\det\begin{pmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{pmatrix} = 0$$

and are as follows:

$$\det\begin{pmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{pmatrix} = (4-\lambda)\cdot(-2-\lambda) - (-1)\cdot 5$$

$$= (\lambda - 4)(\lambda + 2) + 5$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3, -1$$

A. Eigenvectors corresponding to $\lambda = 3$

Plugging our value into $\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

that is $\begin{pmatrix} 4-\lambda & -1 \\ 5 & -2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

gives us $\begin{pmatrix} 4-3 & -1 \\ 5 & -2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & -x_2 \\ 5x_1 & -5x_2 \end{pmatrix} = [x_1 - x_2]\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We see that $x_1 = x_2$, which leads us to the eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where c_1 is a real nonzero number.

B. Eigenvectors corresponding to $\lambda = -1$

Plugging -1 into the matrix gives us this:

$$\begin{pmatrix} 4-(-1) & -1 \\ 5 & -2-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 & -x_2 \\ 5x_1 & -x_2 \end{pmatrix} = [5x_1 - x_2]\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We see that $5x_1 = x_2$, which leads us to the eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ 5c_2 \end{pmatrix} = c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

where c_2 is a real nonzero number.

5. From problem 4:

$$\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}^{-1}$$

6. The linear system of equations $\begin{cases} 4x_1 - 1x_2 = 1 \\ 5x_1 - 2x_2 = -1 \end{cases}$ can be rewritten as follows:

$$\begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Using the methods from problem 1, we are easily able to infer the roots using Cramer's rule.

- $$x_1 = \frac{\det \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}}{\det \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}} = \frac{1 \cdot (-2) - (-1) \cdot (-1)}{-3} = \frac{-3}{-3} = 1$$

- $$x_2 = \frac{\det \begin{pmatrix} 4 & 1 \\ 5 & -1 \end{pmatrix}}{\det \begin{pmatrix} 4 & -1 \\ 5 & -2 \end{pmatrix}} = \frac{4 \cdot (-1) - 1 \cdot 5}{-3} = \frac{-9}{-3} = 3$$

PROBLEM SET 2

1. It looks like the rank of the matrix

$$\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$$

is 3 from inspection, but let's use the following table, just to be sure.

$\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$ Add (–2 times row 1) to row 2 and (–3 times row 1) to row 3.
$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & -7 & 4 \\ 0 & -14 & 2 \end{pmatrix}$ Add (–2 times row 2) to row 3.
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & -7 & 4 \\ 0 & -14 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & -1 \\ 0 & -7 & 4 \\ 0 & 0 & -6 \end{pmatrix}$ Add $\left(\left(-\frac{1}{6}\right)\right.$ times row 3) to row 1 and $\left(\frac{4}{6}\right.$ times row 3) to row 2.
$\begin{pmatrix} 1 & 0 & -\frac{1}{6} \\ 0 & 1 & \frac{4}{6} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -1 \\ 0 & -7 & 4 \\ 0 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ Add $\left(\frac{4}{7}\right.$ times row 2) to row 1.
$\begin{pmatrix} 1 & \frac{4}{7} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

The two matrices $\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ have the same rank, as we saw on pages 196–201.

Since the number of linearly independent vectors among $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -7 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ -6 \end{pmatrix}$ is obviously 3,

the rank of both $\begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -6 \end{pmatrix}$ also must be 3.

Note that the solution is apparent in step three of the table, since triangular $n \times n$ matrices with nonzero main diagonal entries have rank n. This is also true for nonsquare matrices.

2. $\det \begin{pmatrix} 1 & 4 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & -1 \end{pmatrix}$

$= 1 \cdot 1 \cdot (-1) + 4 \cdot 2 \cdot 3 + (-1) \cdot 2 \cdot (-2) - (-1) \cdot 1 \cdot 3 - 4 \cdot 2 \cdot (-1) - 1 \cdot 2 \cdot (-2)$

$= -1 + 24 + 4 + 3 + 8 + 4 = 42$

PROBLEM SET 3

Suppose c is an arbitrary real number.

1. The set is a subspace since both conditions are met.

❶ $$c\begin{pmatrix}\alpha_1\\\beta_1\\5\alpha_1-7\beta_1\end{pmatrix}=\begin{pmatrix}c\alpha_1\\c\beta_1\\5(c\alpha_1)-7(c\beta_1)\end{pmatrix}\in\left\{\begin{pmatrix}\alpha\\\beta\\5\alpha-7\beta\end{pmatrix}\middle|\ \begin{matrix}\alpha\text{ and }\beta\text{ are}\\\text{arbitrary}\\\text{real numbers}\end{matrix}\right\}$$

❷ $$\begin{pmatrix}\alpha_1\\\beta_1\\5\alpha_1-7\beta_1\end{pmatrix}+\begin{pmatrix}\alpha_2\\\beta_2\\5\alpha_2-7\beta_2\end{pmatrix}=\begin{pmatrix}\alpha_1+\alpha_2\\\beta_1+\beta_2\\5(\alpha_1+\alpha_2)-7(\beta_1+\beta_2)\end{pmatrix}\in\left\{\begin{pmatrix}\alpha\\\beta\\5\alpha-7\beta\end{pmatrix}\middle|\ \begin{matrix}\alpha\text{ and }\beta\text{ are}\\\text{arbitrary}\\\text{real numbers}\end{matrix}\right\}$$

2. The set is not a subspace since neither condition is met.[1]

❶ $$2\begin{pmatrix}\alpha_1\\\beta_1\\5\alpha_1-7\end{pmatrix}=\begin{pmatrix}2\alpha_1\\2\beta_1\\5(2\alpha_1)-14\end{pmatrix}\neq\begin{pmatrix}2\alpha_1\\2\beta_1\\5(2\alpha_1)-7\end{pmatrix}\in\left\{\begin{pmatrix}\alpha\\\beta\\5\alpha-7\end{pmatrix}\middle|\ \begin{matrix}\alpha\text{ and }\beta\text{ are}\\\text{arbitrary}\\\text{real numbers}\end{matrix}\right\}$$

❷ $$\begin{pmatrix}\alpha_1\\\beta_1\\5\alpha_1-7\end{pmatrix}+\begin{pmatrix}\alpha_2\\\beta_2\\5\alpha_2-7\end{pmatrix}$$

$$=\begin{pmatrix}\alpha_1+\alpha_2\\\beta_1+\beta_2\\5(\alpha_1+\alpha_2)-14\end{pmatrix}\neq\begin{pmatrix}\alpha_1+\alpha_2\\\beta_1+\beta_2\\5(\alpha_1+\alpha_2)-7\end{pmatrix}\in\left\{\begin{pmatrix}\alpha\\\beta\\5\alpha-7\end{pmatrix}\middle|\ \begin{matrix}\alpha\text{ and }\beta\text{ are}\\\text{arbitrary}\\\text{real numbers}\end{matrix}\right\}$$

1. *Both* conditions on page 151 have to be met for the subset to be a subspace. This means that checking the second condition is unnecessary if we find that the first condition doesn't hold.

PROBLEM SET 4

1. $$\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\right\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\left\|\begin{pmatrix}4\\1\\-2\end{pmatrix}\right\| = \sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{16 + 1 + 4} = \sqrt{21}$$

2. $$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\1\\-2\end{pmatrix} = 1 \cdot 4 + 2 \cdot 1 + 3 \cdot (-2) = 4 + 2 - 6 = 0$$

3. The angle between $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\1\\-2\end{pmatrix}$ can be calculated using the dot product formula as follows:

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\1\\-2\end{pmatrix}}{\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\right\| \cdot \left\|\begin{pmatrix}4\\1\\-2\end{pmatrix}\right\|} = \frac{0}{\sqrt{14} \cdot \sqrt{21}} = 0$$

So the angle is $\cos^{-1} 0 = 90$ degrees.

4. $$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}2 \cdot (-2) & - & 1 \cdot 3\\ 3 \cdot 4 & - & (-2) \cdot 1\\ 1 \cdot 1 & - & 4 \cdot 2\end{pmatrix} = \begin{pmatrix}(-4) - 3\\ 12 + 2\\ 1 - 8\end{pmatrix} = \begin{pmatrix}-7\\14\\-7\end{pmatrix} = 7\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

B
VECTOR SPACES

On page 16 (Chapter 1) it was mentioned that linear algebra is generally about translating something residing in an *m*-dimensional space into a corresponding shape in an *n*-dimensional space. This is by all means true, though understanding a more general interpretation of linear algebra might give you an edge if you decide to study the subject further.

In this interpretation, most of the interesting calculations and theorems have to do with something called *vector spaces*, which are described on the next page. Note that there is a difference between these vectors and the ones presented in Chapter 4—the ones we're discussing here are a much more abstract concept.

The basic idea is this: Much as you play football on football fields and golf on golf courses, you calculate linear algebra in vector spaces.

But before we get into the technical definition of a vector space, let's look at a couple of simple, concrete examples.

EXAMPLE 1

The first example may already be familiar: Let's say that X is the set of all ordered triples of real numbers. So two of the many elements in X are (1.0, 2.3, −4.6) and (0.0, −5.7, 8.1). This infinite set of ordered triples forms a vector space (as described by the axioms listed on the next page). X is a vector space, and (1.0, 2.3, −4.6) is a vector.

EXAMPLE 2

As a second example, consider these two polynomials with real coefficients:

$$7t^4 - 3t - 4 \text{ and } 2t - 1$$

These polynomials could be considered vectors, if we view the set of all polynomials up to the fourth degree as a vector space.

THE EIGHT AXIOMS OF VECTOR SPACES

Assume that $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are elements of the set X, and that c and d are two arbitrary numbers.

If X satisfies the following two sets of axioms, we say that X is a *vector space* and $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are *vectors*.

ADDITION AXIOMS:

The set has to be closed under vector addition. This means that the sum of two elements of the set also belongs to the set.

Vector addition must also satisfy the following four conditions:

1. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ (associativity)
2. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (commutativity)
3. A zero vector ($\mathbf{0}$) exists with the following properties:
   $\boldsymbol{x} + \mathbf{0} = \mathbf{0} + \boldsymbol{x} = \boldsymbol{x}$
4. An inverse vector ($-\boldsymbol{x}$) exists with the following properties:
   $\boldsymbol{x} + (-\boldsymbol{x}) = (-\boldsymbol{x}) + \boldsymbol{x} = \mathbf{0}$

SCALAR MULTIPLICATION AXIOMS:

The set has to be closed under scalar multiplication. This means that the product of an element of the set and an arbitrary number also belongs to the set.

Scalar multiplication must also satisfy the following four conditions:

5. $c(\boldsymbol{x} + \boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$
6. $(cd)\boldsymbol{x} = c(d\boldsymbol{x})$
7. $(c + d)\boldsymbol{x} = c\boldsymbol{x} + d\boldsymbol{x}$
8. $1\boldsymbol{x} = \boldsymbol{x}$

In this book we always assume that scalar multiplication is done with real numbers. Such vector spaces are usually called *real vector spaces*. Vector spaces also allowing multiplication with complex numbers would similarly be called *complex vector spaces*.

C

DOT PRODUCT

NORM

Suppose we have an arbitrary vector in R^n $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

The vector *norm* or *length* is then equal to $\sqrt{x_1{}^2 + x_2{}^2 + \dots + x_n{}^2}$

and is written $\left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\|$.

EXAMPLE 1

$$\left\| \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \right\| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$$

EXAMPLE 2

$$\left\| \begin{pmatrix} \sqrt{2} - \sqrt{6} \\ \sqrt{2} + \sqrt{6} \end{pmatrix} \right\| = \sqrt{(\sqrt{2} - \sqrt{6})^2 + (\sqrt{2} + \sqrt{6})^2} = \sqrt{2 - 2\sqrt{12} + 6 + 2 + 2\sqrt{12} + 6} = \sqrt{16} = 4$$

DOT PRODUCT

Suppose we have two arbitrary vectors $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in R^n.

The vector's *dot product*[1] is defined as follows:

$$x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

This is usually represented with a dot (·) like so:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

EXAMPLE

$$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} - \sqrt{6} \\ \sqrt{2} + \sqrt{6} \end{pmatrix} = 1 \cdot (\sqrt{2} - \sqrt{6}) + \sqrt{3} \cdot (\sqrt{2} + \sqrt{6}) = \sqrt{2} - \sqrt{6} + \sqrt{6} + \sqrt{18} = \sqrt{2} + 3\sqrt{2} = 4\sqrt{2}$$

1. The dot product is sometimes referred to as the *scalar product*.

THE ANGLE BETWEEN TWO VECTORS

Suppose we have two arbitrary vectors $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ in R^n.

The angle θ between those two vectors can be found using the following relationship:

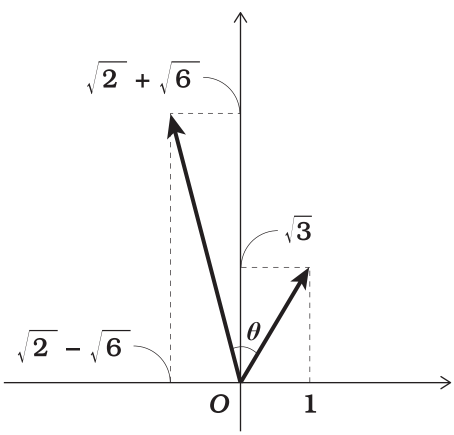
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\| \cdot \cos\theta$$

EXAMPLE

The angle θ between the two vectors $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{2} - \sqrt{6} \\ \sqrt{2} + \sqrt{6} \end{pmatrix}$ can be found using the formula:

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} - \sqrt{6} \\ \sqrt{2} + \sqrt{6} \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} \sqrt{2} - \sqrt{6} \\ \sqrt{2} + \sqrt{6} \end{pmatrix} \right\|} = \frac{4\sqrt{2}}{2 \cdot 4} = \frac{\sqrt{2}}{2}$$

So θ = 45 degrees.



INNER PRODUCTS

The dot product is actually a special case of a more general concept that has some very interesting applications. That general concept is a function, called an *inner product*, that maps two vectors to a real number and also satisfies some special properties. There are also *inner product spaces*,[2] which are vector spaces that have an associated inner product, as described below.

REAL INNER PRODUCT SPACES

We say that the real vector space X is a *real inner product space* or *Euclidean space* if there exists a real inner product $<\boldsymbol{x}, \boldsymbol{y}>$ which maps a pair of vectors to a scalar and satisfies the following conditions for all vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, and all scalars c:

❶ $<\boldsymbol{x}, \boldsymbol{y}> = <\boldsymbol{y}, \boldsymbol{x}>$

❷ $<c\boldsymbol{x}, \boldsymbol{y}> = c<\boldsymbol{x}, \boldsymbol{y}> = <\boldsymbol{x}, c\boldsymbol{y}>$

❸ $<\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z}> = <\boldsymbol{x}, \boldsymbol{y}> + <\boldsymbol{x}, \boldsymbol{z}>$ and $<\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}> = <\boldsymbol{x}, \boldsymbol{z}> + <\boldsymbol{y}, \boldsymbol{z}>$

❹ $<\boldsymbol{x}, \boldsymbol{x}> \geq \mathbf{0}$ and $<\boldsymbol{x}, \boldsymbol{x}> = \mathbf{0}$ only when $\boldsymbol{x} = \mathbf{0}$ (the zero vector).

The dot product is the most familiar example of an inner product. In that example, we define

$$<\boldsymbol{x}, \boldsymbol{y}> = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

2. The subject is outside the scope of this book, but inner products also appear in complex product spaces.

Orthonormal Bases

Vector sets like

$$\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\} \quad \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\right\} \text{ and } \left\{\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix},\ \frac{1}{\sqrt{21}}\begin{pmatrix}4\\1\\-2\end{pmatrix},\ \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\}$$

where

- the norm of every vector is equal to 1
- the dot product of each vector pair is equal to 0

are called *orthonormal bases* or *ON-bases*.

The Gram-Schmidt orthogonalization process can be used to create an orthonormal basis from any arbitrary basis, but it is outside the scope of this book.

D

CROSS PRODUCT

WHAT IS THE CROSS PRODUCT?

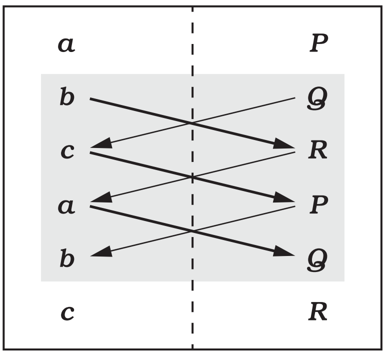
Suppose we have two arbitrary vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ in R^3.

The vector cross product is defined as $\begin{pmatrix} bR - Qc \\ cP - Ra \\ aQ - Pb \end{pmatrix}$

and is usually represented with a cross $\times$ like so: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$

NOTE *The cross product is defined only in* R^3*. In contrast, the dot product is defined in* R^{n} *for all positive* n.

Here's a good mnemonic for remembering the combinations in calculating the cross product of two vectors:



Start by writing the elements of each vector twice, as you can see above. Ignoring the first and last rows, draw an arrow from each element to the one below it in the opposite vector.

Arrows going from left to right get a plus sign; arrows going from right to left get a minus sign. The top pair of arrows produces the first component of the cross product, the middle pair produces the second component, and the bottom pair produces the last component.

CROSS PRODUCT AND PARALLELOGRAMS

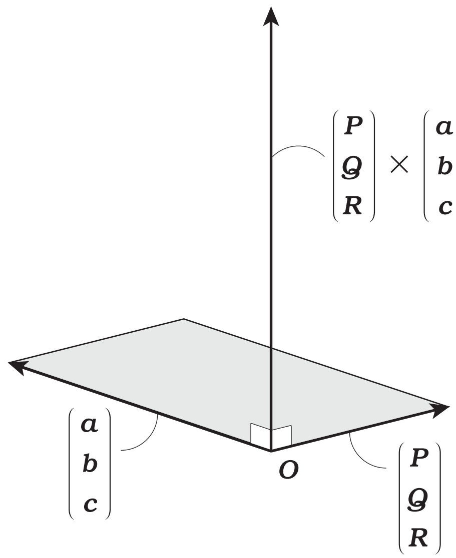
Consider the following cross product:

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

❶ It is perpendicular to both vectors $\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

❷ Its length is equal to the area of the parallelogram with sides $\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Both properties are illustrated in the picture below.



NOTE *This picture is using a "right handed" coordinate system. That means that your right thumb will point in the direction of the cross product if you do the following: Stick your thumb out so it is perpendicular to your lower arm, then use your remaining four fingers to form the letter* C. *Starting with the base of your fingers as the vector on the left side of the cross product, orient your hand so the tips of your fingers are pointing toward the vector on the right side of the cross product. Your thumb will then be pointing in the direction of the result of the cross product! Note that if you switch the positions of the vectors, the cross product will reverse direction.*

Let's make sure that both ❶ and ❷ hold.

❶

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \right) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} bR - Qc \\ cP - Ra \\ aQ - Pb \end{pmatrix}$$

$$= a(bR - Qc) + b(cP - Ra) + c(aQ - Pb)$$

$$= abR - aQc + bcP - bRa + caQ - cPb$$

$$= 0$$

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \right) = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} bR - Qc \\ cP - Ra \\ aQ - Pb \end{pmatrix}$$

$$= P(bR - Qc) + Q(cP - Ra) + R(aQ - Pb)$$

$$= PbR - PQc + QcP - QRa + RaQ - RPb$$

$$= 0$$

❷

$$\left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} bR - Qc \\ cP - Ra \\ aQ - Pb \end{pmatrix} \right\|^2$$

θ is the angle between $\begin{pmatrix} P \\ Q \\ R \end{pmatrix}$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$= (bR - Qc)^2 + (cP - Ra)^2 + (aQ - Pb)^2$$

$$= (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2) - (aP + bQ + cR)^2$$

$$= (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2) - \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \right)^2$$

$$= (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2) - (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2) \cos^2\theta$$

$$= (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2)(1 - \cos^2\theta)$$

$$= (a^2 + b^2 + c^2)(P^2 + Q^2 + R^2) \sin^2\theta$$

$$= \left(\left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\| \left\| \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \right\| \sin\theta \right)^2$$

CROSS PRODUCT AND DOT PRODUCT

The table below contains a comparison between cross and dot products.

CROSS PRODUCT	DOT PRODUCT
$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}2\cdot 6-5\cdot 3\\3\cdot 4-6\cdot 1\\1\cdot 5-4\cdot 2\end{pmatrix}$ $= -\begin{pmatrix}5\cdot 3-2\cdot 6\\6\cdot 1-3\cdot 4\\4\cdot 2-1\cdot 5\end{pmatrix} = -\begin{pmatrix}4\\5\\6\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} = 1\cdot 4+2\cdot 5+3\cdot 6$ $= 4\cdot 1+5\cdot 2+6\cdot 3 = \begin{pmatrix}4\\5\\6\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}$
$\begin{pmatrix}1c\\2c\\3c\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}2c\cdot 6-5\cdot 3c\\3c\cdot 4-6\cdot 1c\\1c\cdot 5-4\cdot 2c\end{pmatrix}$ $= c\begin{pmatrix}2\cdot 6-5\cdot 3\\3\cdot 4-6\cdot 1\\1\cdot 5-4\cdot 2\end{pmatrix} = c\left(\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix}\right)$	$\begin{pmatrix}1c\\2c\\3c\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} = 1c\cdot 4+2c\cdot 5+3c\cdot 6$ $= c\,(1\cdot 4+2\cdot 5+3\cdot 6) = c\left(\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix}\right)$
$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \left(\begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}7\\8\\9\end{pmatrix}\right)$ $= \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4+7\\5+8\\6+9\end{pmatrix}$ $= \begin{pmatrix}2\cdot(6+9)-(5+8)\cdot 3\\3\cdot(4+7)-(6+9)\cdot 1\\1\cdot(5+8)-(4+7)\cdot 2\end{pmatrix}$ $= \begin{pmatrix}2\cdot 6-5\cdot 3\\3\cdot 4-6\cdot 1\\1\cdot 5-4\cdot 2\end{pmatrix} + \begin{pmatrix}2\cdot 9-8\cdot 3\\3\cdot 7-9\cdot 1\\1\cdot 8-7\cdot 2\end{pmatrix}$ $= \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}7\\8\\9\end{pmatrix}$	$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \left(\begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}7\\8\\9\end{pmatrix}\right)$ $= \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4+7\\5+8\\6+9\end{pmatrix}$ $= 1\cdot(4+7)+2\cdot(5+8)+3\cdot(6+9)$ $= (1\cdot 4+2\cdot 5+3\cdot 6)+(1\cdot 7+2\cdot 8+3\cdot 9)$ $= \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}7\\8\\9\end{pmatrix}$

E

USEFUL PROPERTIES OF DETERMINANTS

Determinants have several interesting properties. We'll look at seven of them in this appendix.

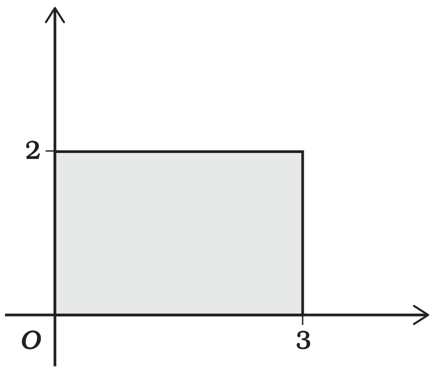
PROPERTY 1

For any square matrix A, $\det A = \det A^T$.

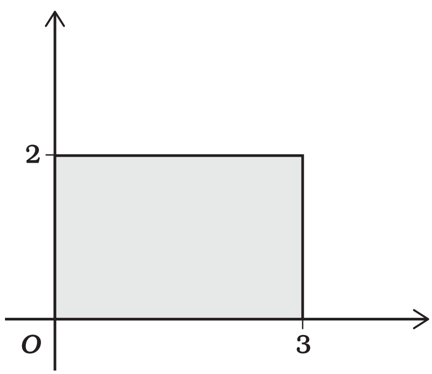
$$\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^T$$

EXAMPLE

- $\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



- $\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}^T = \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



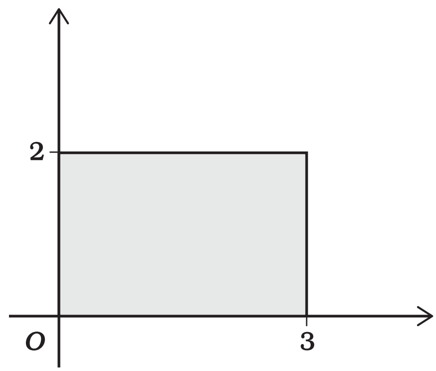
PROPERTY 2

If two columns or two rows of A are interchanged, resulting in matrix B, then det B = −det A.

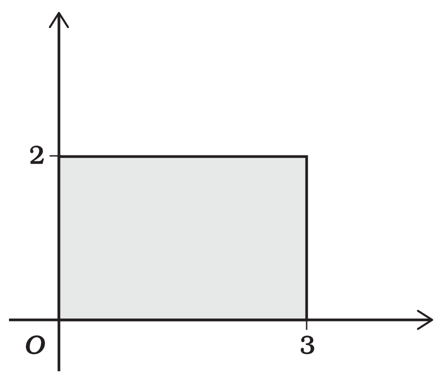
$$\det\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} = (-1)\det\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix}$$

EXAMPLE

- $\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



- $(-1)\det\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} = (-1) \cdot (-6) = 6$



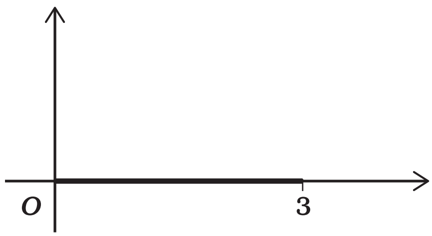
PROPERTY 3

If A has two identical columns or two identical rows, then $\det A = 0$.

$$\det \begin{array}{c} \left(\begin{array}{ccccccc} a_{11} & \cdots & b_1 & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & b_n & \cdots & a_{nn} \end{array}\right) \\ \begin{array}{ccccccc} & & \text{column } i & & \text{column } j & & \end{array} \end{array} = 0$$

EXAMPLE

- $\det \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} = 0$



The area is equal to zero.

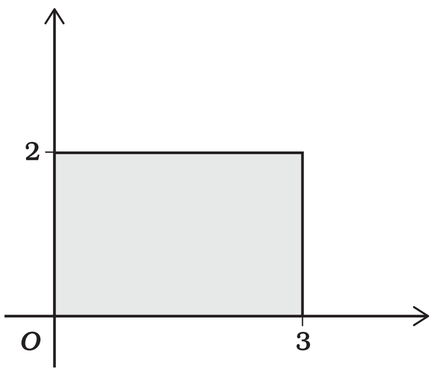
PROPERTY 4

If a column or row of A is multiplied by the constant c, resulting in matrix B, then $\det B = c \det A$, or equivalently, $\det A = \frac{1}{c} \det B$.

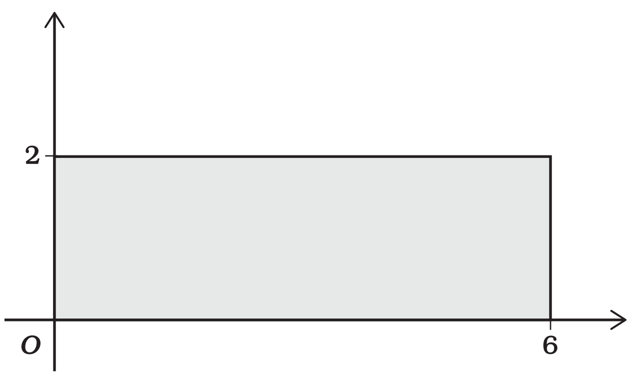
$$\det \begin{pmatrix} a_{11} & \cdots & a_{1i} \cdot c & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} \cdot c & \cdots & a_{nn} \end{pmatrix} = c \det \begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix}$$

EXAMPLE

- $\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



- $\det \begin{pmatrix} 3 \cdot 2 & 0 \\ 0 \cdot 2 & 2 \end{pmatrix} = \det \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = 2 \cdot 6 = 2 \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$



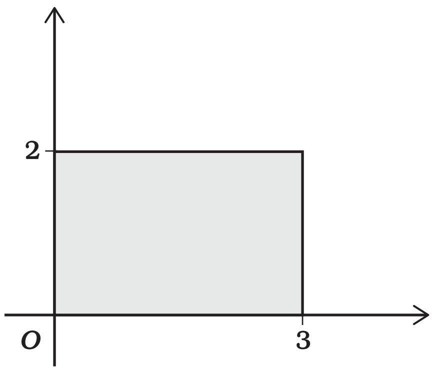
PROPERTY 5

Let A and B be identical square matrices except that the ith columns (or ith rows) differ. Let C be a matrix that is identical to A and B except that the ith column (or ith row) of C is the sum of the ith columns (or ith rows) of A and B. Then $\det C = \det A + \det B$.

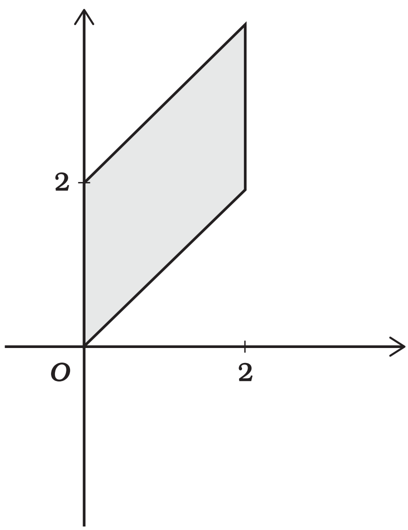
$$\det\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{pmatrix} + \det\begin{pmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{pmatrix} = \det\begin{pmatrix} a_{11} & \cdots & a_{1i}+b_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni}+b_{ni} & \cdots & a_{nn} \end{pmatrix}$$

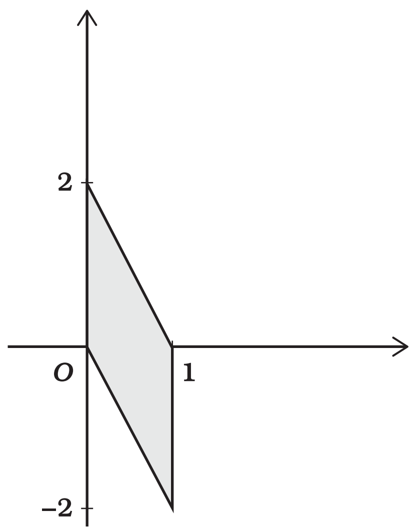
EXAMPLE

- $\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



- $\det\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = \det\begin{pmatrix} 2+1 & 0 \\ 2-2 & 2 \end{pmatrix} = \det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$





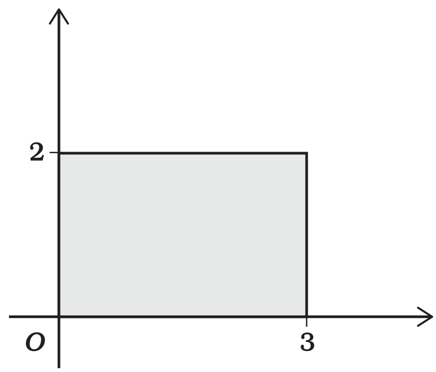
PROPERTY 6

Let B be the matrix formed by replacing column j (or row j) of A with the sum of column j (or row j) of A and a nonzero multiple, c, of column i (or row i) of A, where $i \neq j$. Then $\det B = \det A$.

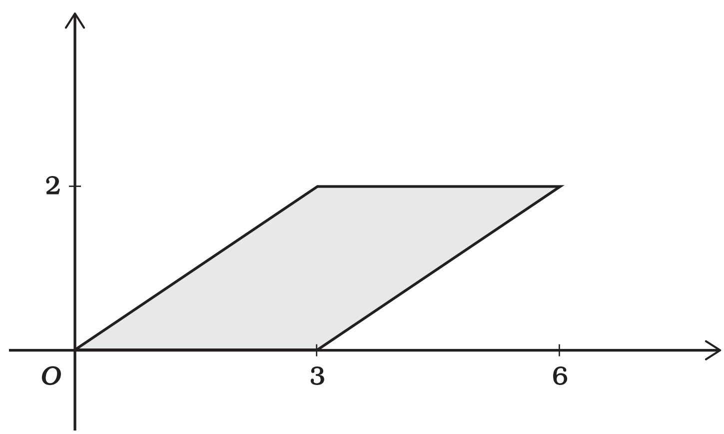
$$\det\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} = \det\begin{pmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1j} + (a_{1i} \cdot c) & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nj} + (a_{ni} \cdot c) & \cdots & a_{nn} \end{pmatrix}$$

EXAMPLE

- $\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 6$



- $\det\begin{pmatrix} 3 & 0 + (3 \cdot 1) \\ 0 & 2 + (0 \cdot 1) \end{pmatrix} = \det\begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix} = 6$



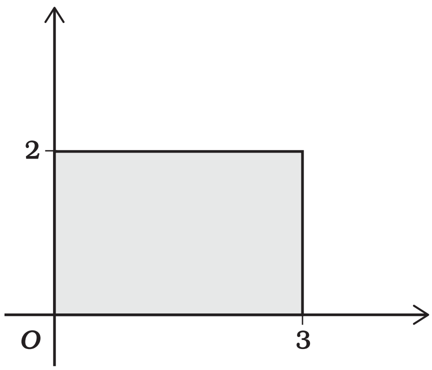
PROPERTY 7

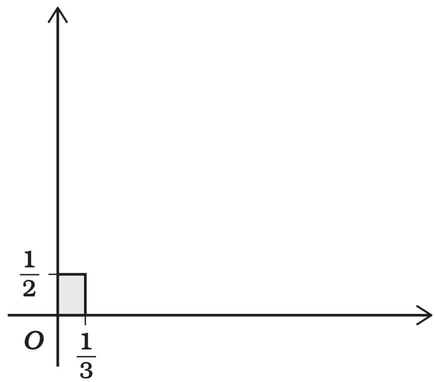
Let A and B be any two square matrices. Then $(\det A)(\det B) = \det(AB)$.

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \det\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \det\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}\right]$$

EXAMPLE

- $\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \cdot \det\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 6 \cdot \frac{1}{6} = 1$





- $\det\left[\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right] = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$